Adversarial training should be cast as a non-zero sum game

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Foundations of AI Seminar Series

Laboratory for Information and Inference Systems (LIONS)
École Polytechnique Fédérale de Lausanne (EPFL)
Switzerland

The University of Warwick

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- Many talented collaborators

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Preface: A new landscape for research

Artificial Intelligence

Machine Learning

Deep Learning

Generative AI

LLM/VLMs

GPT-X

...

Preface: A new landscape for research

- My research:
  - Optimization
  - Deep Learning
  - Reinforcement Learning
- My current courses:
  - Mathematics of Data
  - Reinforcement Learning
  - Online Learning in Games
  - Advanced Topics in ML

Artificial Intelligence

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...

GenAI as an example

GenAI as an example

GenAI as an example

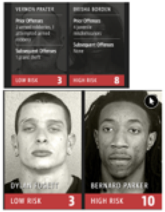
GenAI as an example

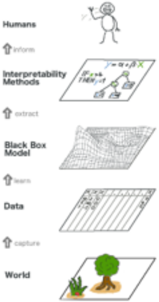
Let's work backwards with GenAI as a running example

- What do customers want?

- What do customers care about?
  - Response speed (inference), availability, cost...
  - Quality of the answers (correct, fair, unbiased, aligned, robust,...)
  - Personalization, privacy,...

The loop now works ... but many challenges A-RISE



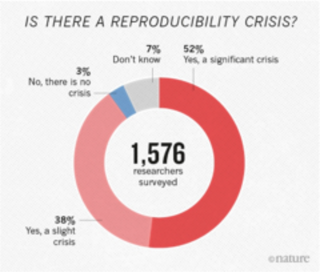


GPT-3 medical chatbot tells suicidal test patient to kill themselves

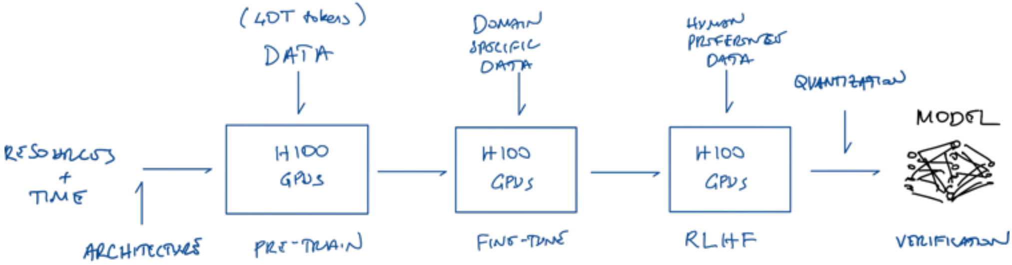
Rob Beschizza — 2 days ago

Researchers experimenting with GPT-3, the AI text-generation model, found that it is not ready to replace human respondents in the chatbox.

1. Robustness
2. Interpretability
3. Bias & Fairness
4. Reproducibility



Research@LIONS: Theory and Methodology



- Optimization: Scalable robust/ distributed/federated/game theoretic, limits of algorithms, online
- Deep learning: Sample complexity, architecture design, optimization formulations
- GenAI: GANs, Langevin Dynamics (e.g., diffusion models), mixture of expert models
- Reinforcement learning: Inverse RL, imitation learning, robust RL
- Trust but verify: Lipschitz constant estimation, decision verification

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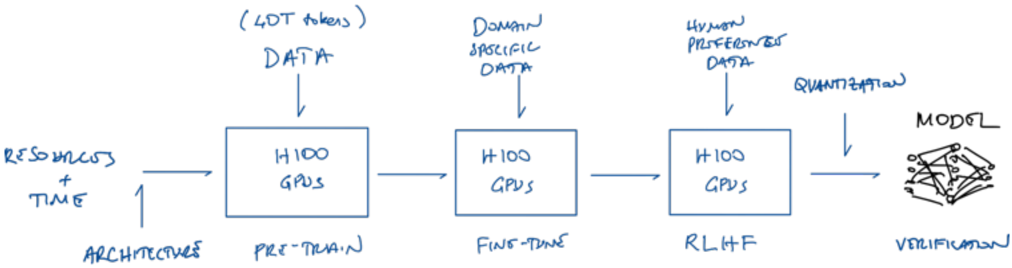

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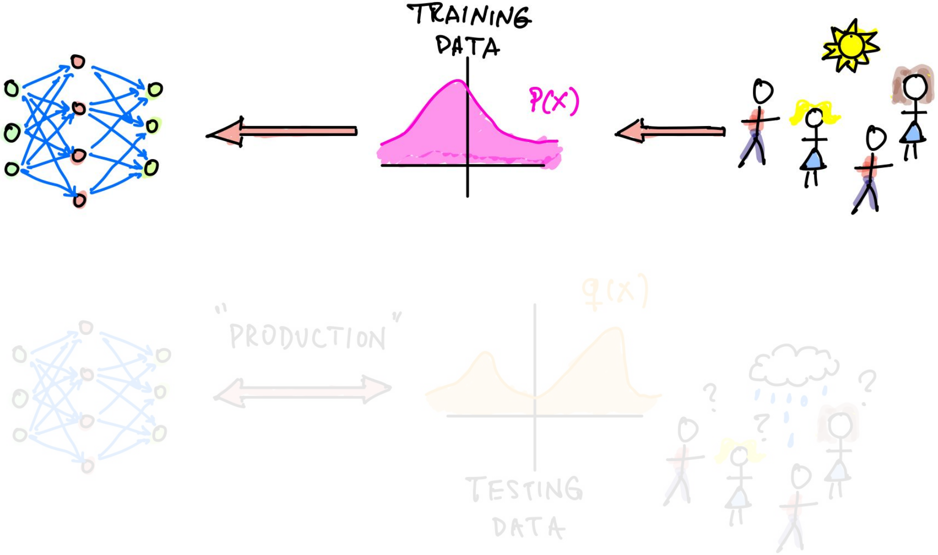
Highlight: Robustness

Why do we need robustness?



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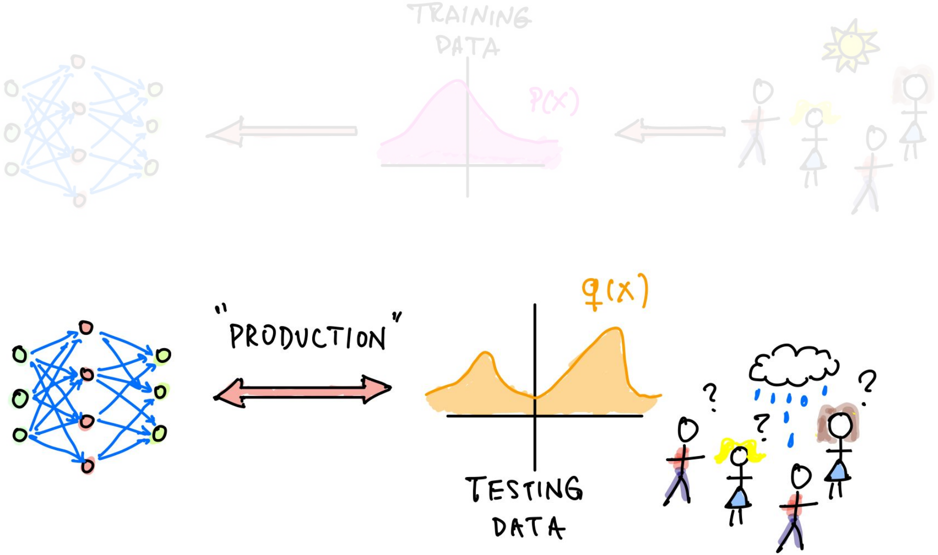
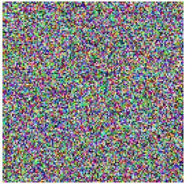


Robustness meets the adversaries

+0.01x



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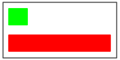

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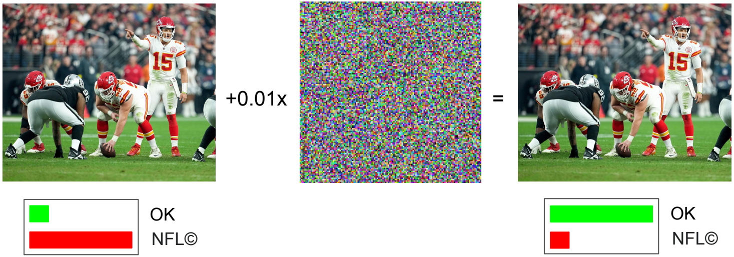
Robustness meets the adversaries

Yuan et al. (2019)

Saadatpanah, Shafahi and Goldstein. (ICML 2020)



Today: “Basic” robust machine learning

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$$

- A seemingly simple optimization formulation
- Critical in machine learning with many applications
  - Adversarial examples and training
  - Generative adversarial networks
  - Robust reinforcement learning

Warm up: Flexibility of the template

$$\Phi^\star = \min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}) \quad (\text{argmin}, \text{argmax} \to \mathbf{x}^\star, \mathbf{y}^\star)$$

Warm up: Flexibility of the template

$$\Phi^\star = \min_{\mathbf{x}\in\mathcal{X}} \underbrace{\max_{\mathbf{y}:\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})}_{f(\mathbf{x})} \quad (\text{argmin}, \text{argmax} \to \mathbf{x}^\star, \mathbf{y}^\star)$$

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- (eula) In the sequel,
  - the set $\mathcal{X}$ is convex
  - all convergence characterizations are with feasible iterates $\mathbf{x}^k \in \mathcal{X}$
  - L-smooth means $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x}-\mathbf{y}\|, \forall \mathbf{x},\mathbf{y} \in \mathcal{X}$
  - ∇ may refer to the generalized subdifferential

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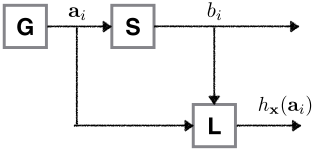
A deep learning optimization problem in supervised learning



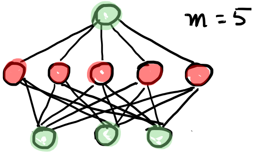
Definition (Optimization formulation)

The "deep-learning" problem with a neural network $h_{\mathbf{x}}(\mathbf{a})$ is given by

$$\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathcal{X}} \left\{ f(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} L(h_{\mathbf{x}}(\mathbf{a}_i), b_i) \right\},$$

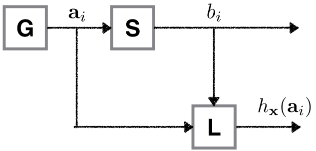
where $\mathcal{X}$ denotes the constraints and L is a loss function.

- A single hidden layer neural network with params $\mathbf{x} := [\mathbf{X}_1, \mathbf{X}_2, \mu_1, \mu_2]$



$$h_{\mathbf{x}}(\mathbf{a}) := \underbrace{\Big[\ \mathbf{X}_2\ \Big]\ \overset{\text{activation}}{\sigma}\left(\Big[\ \overset{\text{weight}}{\mathbf{X}_1}\ \Big] \Big[\overset{\text{input}}{\mathbf{a}}\Big] + \Big[\overset{\text{bias}}{\mu_1}\Big] \right)}_{\text{hidden layer = learned features}} + \Big[\overset{\text{bias}}{\mu_2}\Big]$$

A deep learning optimization problem in supervised learning



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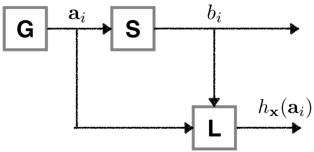
Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\boldsymbol{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with $\boldsymbol{a}_i \in \mathbb{R}^p$ and $\mathbf{b}_i$ be the corresponding labels. The adversarial training optimization problem is given by

$$\min_{\mathbf{x}} \left\{ \frac{1}{n}\sum_{i=1}^{n} f_i(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} \underbrace{\left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\le\epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}), \mathbf{b}_i) \right]}_{=: f_i(\mathbf{x})} \right\}.$$

Note that L is not continuously differentiable due to ReLU, max-pooling, etc.

A deep learning optimization problem in supervised learning



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Example objectives in different tasks

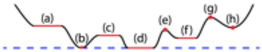
- $\min_{\mathbf{x}} \left\{ \frac{1}{n}\sum_{i=1}^{n} \left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|_\infty \le \epsilon} L\left(h_{\mathbf{x}}\left(\mathbf{a}_i + \boldsymbol{\delta}\right), \mathbf{b}_i\right) \right] \right\}$ Adversarial training [11].
- $\min_{\mathbf{x}} \left\{ \frac{1}{n}\sum_{i=1}^{n} \left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|_2 \le \epsilon} L(h_{\mathbf{x}+\boldsymbol{\delta}}\left(\mathbf{a}_i\right), \mathbf{b}_i) \right] \right\}$ ϵ-stability training [3], Sharpness-aware minimization [7].
- $\min_{\mathbf{x}} \max_{\mathbf{b}^c \in [C]} \frac{1}{n_c} \sum_{i=1}^{n_c} \left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\| \le \epsilon} L\left(h_{\mathbf{x}}\left(\mathbf{a}_i + \boldsymbol{\delta}\right), \mathbf{b}_i^c\right) \right]$ Class fairness [16].

Basic questions on solution concepts

- Consider the finite sum setting

$$f^\star := \min_{\mathbf{x}\in\mathbb{R}^p} \left\{ f(\mathbf{x}) := \frac{1}{n}\sum_{j=1}^{n} f_j(\mathbf{x}) \right\}.$$

- Goal: Find $\mathbf{x}^\star$ such that $\nabla f(\mathbf{x}^\star) = 0$.



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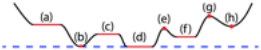

Vanilla (Minibatch) SGD
Input: Stochastic gradient oracle $\mathbf{g}$, initial point $\boldsymbol{x}^0$, step size α_k
1. For $k = 0, 1, \ldots$:   obtain the (minibatch) stochastic gradient $\mathbf{g}^k$ update $\mathbf{x}^{k+1} \leftarrow \mathbf{x}^k - \gamma_k \mathbf{g}^k$

Solving the outer problem: Gradient computation

Adversarial Training

Let $h_{\mathbf{x}} : \mathbb{R}^p \to \mathbb{R}$ be a model with parameters $\mathbf{x}$ and let $\{(\boldsymbol{a}_i, \mathbf{b}_i)\}_{i=1}^n$, with $\boldsymbol{a}_i \in \mathbb{R}^p$ and $\mathbf{b}_i$ be the corresponding labels. The adversarial training optimization problem is given by

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Note that L is not continuously differentiable due to ReLU, max-pooling, etc.

Question

How can we compute the following stochastic gradient (i.e., $\mathbb{E}_i \nabla_{\mathbf{x}} f_i(\mathbf{x}) = \nabla_{\mathbf{x}} f_i(\mathbf{x})$ for $i \sim \text{Uniform}\{1, \ldots, n\}$):

$$\nabla_{\mathbf{x}} f_i(\mathbf{x}) := \nabla_{\mathbf{x}} \left(\max_{\boldsymbol{\delta} : \|\boldsymbol{\delta}\| \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}), \mathbf{b}_i) \right)?$$

◦ **Challenge:** It involves differentiating with respect to a maximization.

Danskin's Theorem (1966): How do we compute the gradient?

Theorem ([5])

Let $\mathcal{S}$ be compact set, $\Phi : \mathbb{R}^p \times \mathcal{S}$ be continuous such that $\Phi(\cdot, \mathbf{y})$ is differentiable for all $\mathbf{y} \in \mathcal{S}$, and $\nabla_{\mathbf{x}}\Phi(\mathbf{x}, \mathbf{y})$ be continuous on $\mathbb{R}^p \times \mathcal{S}$. Define

$$f(\mathbf{x}) := \max_{\mathbf{y}\in\mathcal{S}} \Phi(\mathbf{x}, \mathbf{y}), \qquad \mathcal{S}^\star(\mathbf{x}) := \arg\max_{\mathbf{y}\in\mathcal{S}} \Phi(\mathbf{x}, \mathbf{y}).$$

Let $\gamma \in \mathbb{R}^p$, and $\|\gamma\|_2 = 1$. The directional derivative $D_\gamma f(\bar{\mathbf{x}})$ of f in the direction γ at $\bar{\mathbf{x}}$ is given by

$$D_\gamma f(\bar{\mathbf{x}}) = \max_{\mathbf{y}\in\mathcal{S}^\star(\bar{\mathbf{x}})} \langle \gamma, \nabla_{\mathbf{x}}\Phi(\bar{\mathbf{x}}, \mathbf{y})\rangle.$$

An immediate consequence

If $\boldsymbol{\delta}^\star \in \arg\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}), \mathbf{b}_i)$ is unique, then we have

$$\nabla_{\mathbf{x}} f_i(\mathbf{x}) = \nabla_{\mathbf{x}} L(h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}^\star), \mathbf{b}_i).$$

Optimized perturbations are typically not unique!

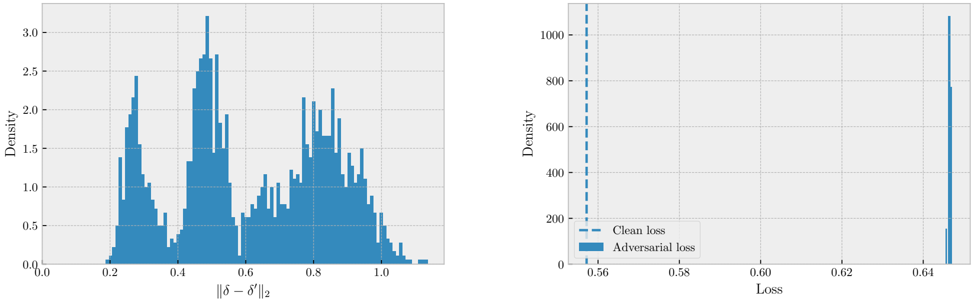

Figure: (*left*) Pairwise ℓ_2-distances between "optimized" perturbations with different initializations are bounded away from zero. (*right*) The losses of multiple perturbations on the same sample concentrate around a value much larger than the clean loss.

Theoretical foundations

	unique $\delta^\star$	non-unique $\delta^\star$
$\nabla_{\mathbf{x}}\Phi(\mathbf{x},\delta^\star)$	$\nabla_{\mathbf{x}} f(\mathbf{x})$	descent direction [13]

Published as a conference paper at ICLR 2018

TOWARDS DEEP LEARNING MODELS RESISTANT TO ADVERSARIAL ATTACKS

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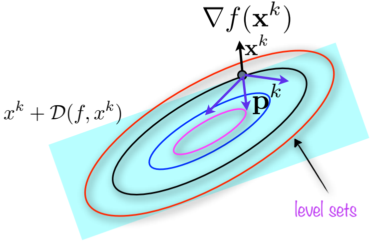

Theoretical foundations ?

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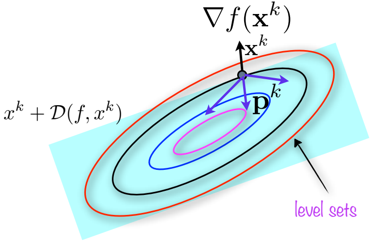
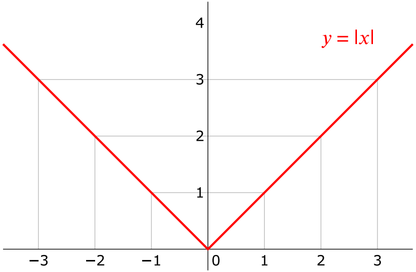


A counterexample

$$f(\mathbf{x}) := \max_{\boldsymbol{\delta}\in[-1,1]} \mathbf{x}\boldsymbol{\delta} = |\mathbf{x}|\,.$$



- We have $\mathcal{S} := [-1,1]$ and $\Phi(\mathbf{x},\boldsymbol{\delta}) = \mathbf{x}\boldsymbol{\delta}$.
- At $\mathbf{x} = 0$, we have $\mathcal{S}^\star(0) = [-1,1]$.
- We can choose $\delta = 1 \in \mathcal{S}^\star(0)$: $\Phi(\mathbf{x},1) = \mathbf{x}$.

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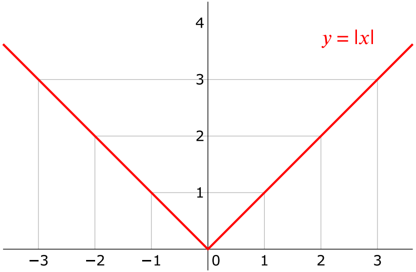

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- We can choose $\delta = 1 \in \mathcal{S}^\star(0)$: $\Phi(\mathbf{x}, 1) = \mathbf{x}$.
  - $-\nabla_{\mathbf{x}} \Phi(0, 1) = -1 \neq 0$.
  - Is -1 a descent direction at $\mathbf{x} = 0$?

Our understanding [Latorre, Krawczuk, Dadi, Pethick, Cevher, ICLR (2023)]

- The corollary in [13] is false (it is subtle!).
- We constructed a counter example & proposed an alternative way (DDi) of computing "the gradient":

	unique $\delta^\star$	non-unique $\delta^\star$
$\nabla_{\mathbf{x}}\Phi(\mathbf{x},\delta^\star)$	$\nabla_{\mathbf{x}} f(\mathbf{x})$	could be ascent direction!

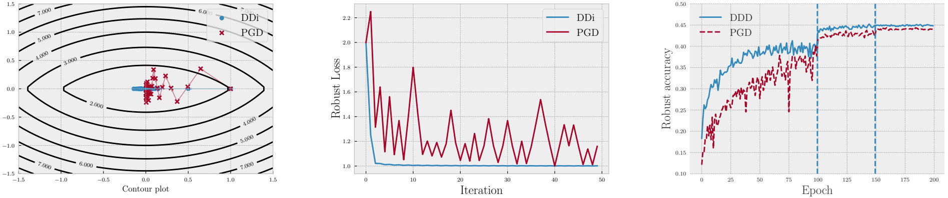
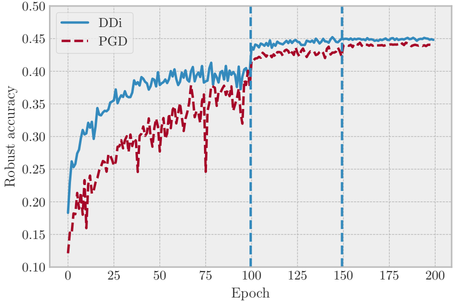


Figure: Left and middle pane: comparison DDi and PGD ([13]) on a synthetic problem. Right pane: DDi vs PGD on CIFAR10.

Comparison with the state-of-the-art



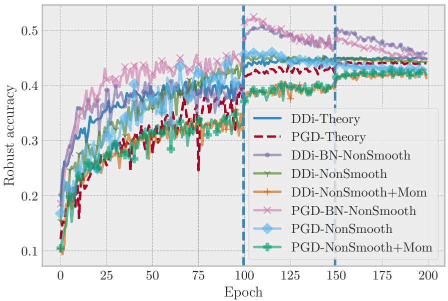
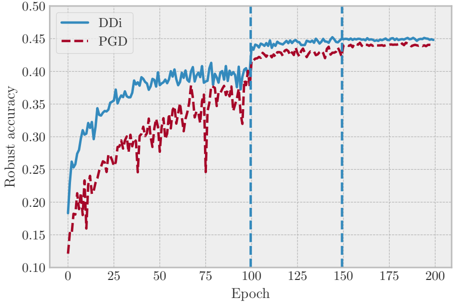
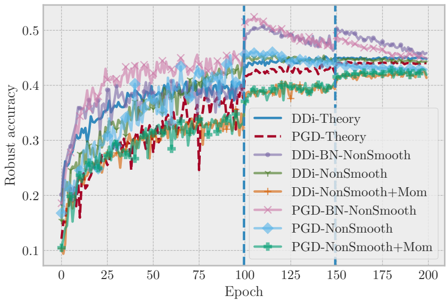


Figure: (left) PGD vs DDi on CIFAR10, in a setting covered by theory. (right) An ablation testing the effect of adding back the elements not covered by theory (BN,ReLU,momentum).

Comparison with the state-of-the-art





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DDi + Graduate Student Descent may improve things (performance or catastrophic overfitting)?

Out of the frying pan into the fire

Original Formulation of Adversarial Training (I)

$$\min_{\mathbf{x}} \mathbb{E} \left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\| \leq \epsilon} L(h_{\mathbf{x}}(\mathbf{a} + \boldsymbol{\delta}), b) \right]$$

Original Formulation of Adversarial Training (I)

$$\min_{\mathbf{x}} \mathbb{E}\left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b)\right]$$

which loss L?

Original Formulation of Adversarial Training (II)

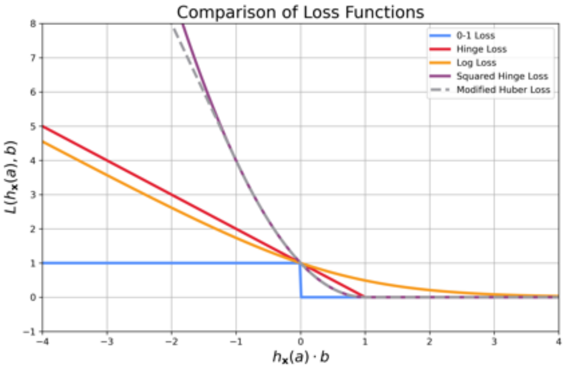
$$\min_{\mathbf{x}} \mathbb{E}\left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L_{01}(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b)\right]$$

Original Formulation of Adversarial Training (II)

$$\min_{\mathbf{x}} \mathbb{E}\left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L_{01}(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b)\right]$$

$$\min_{\mathbf{x}} \mathbb{E}\left[\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L_{\mathsf{CE}}(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b)\right]$$

Surrogate-based optimization for Risk Minimization



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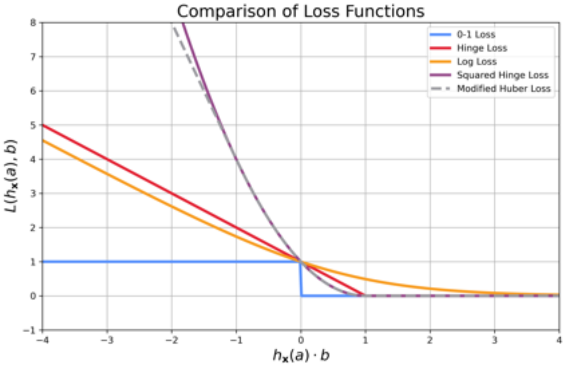
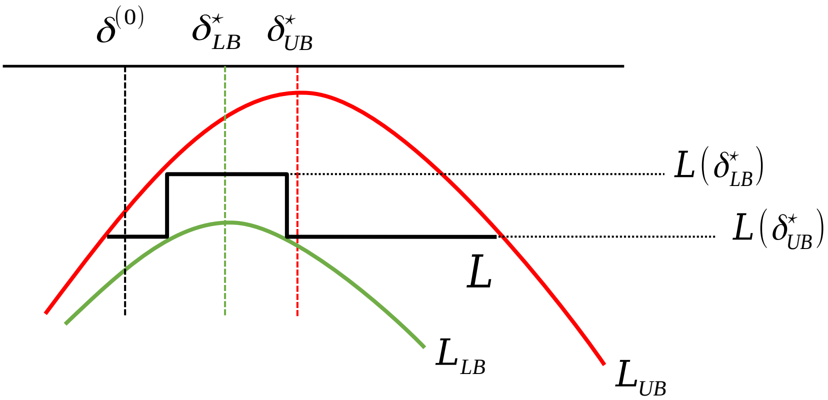


$$\mathbb{E}\left[L_{01}(h_{\mathbf{x}^\star}(\mathbf{a}+\boldsymbol{\delta}), b)\right] \leq \min_{\mathbf{x}} \mathbb{E}\left[L_{\mathsf{CE}}(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b)\right]$$

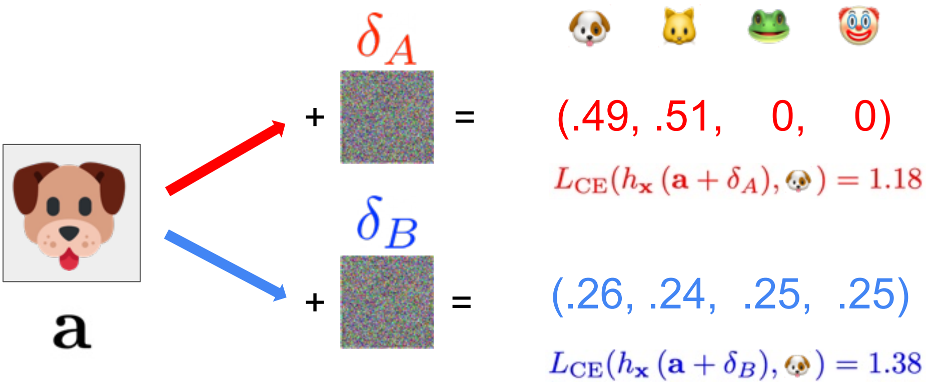
Adversary maximizes an upper bound (I)

$$L_{01}\left(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}^{\star}), b\right) \leq \max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L_{\mathsf{CE}}\left(h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta}), b\right)$$

Adversary maximizes an upper bound (II)



Why maximizing cross-entropy leads to weak adversaries



Adversary's problem can be "solved" without using surrogates

Theorem (Reformulation of the Adversary's problem)

$$\boldsymbol{\delta}^\star \in \arg\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} \max_{j\neq\mathbf{b}} h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta})_j - h_{\mathbf{x}}(\mathbf{a}+\boldsymbol{\delta})_{\mathbf{b}} \Rightarrow$$

$$\boldsymbol{\delta}^\star \in \arg\max_{\boldsymbol{\delta}:\|\boldsymbol{\delta}\|\leq\epsilon} L_{01}(\mathbf{x}, \mathbf{a}+\boldsymbol{\delta}, \mathbf{b})$$

Bilevel Optimization [Robey,* Latorre,* Pappas, Hassani, Cevher(2023)][1]

- Best targeted attack (BETA) optimization formulation:

$$\min_{\mathbf{x}\in\mathbf{X}} \frac{1}{n}\sum_{i=1}^{n} L_{\mathsf{CE}}(\mathbf{x}, \mathbf{a}_i + \boldsymbol{\delta}^\star_{i,j^\star}, \mathbf{b}_i)$$

$$\text{such that } \boldsymbol{\delta}^\star_{i,j} \in \arg\max_{\boldsymbol{\delta}:\ \|\boldsymbol{\delta}\|\leq\epsilon} h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta})_j - h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta})_{\mathbf{b}_i}$$

$$j^\star \in \arg\max_{j\in[K]-\{\mathbf{b}_i\}} h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}_{i,j^\star})_j - h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}_{i,j^\star})_{\mathbf{b}_i}$$

[1] https://infoscience.epfl.ch/record/302995 or https://tinyurl.com/33yup77v

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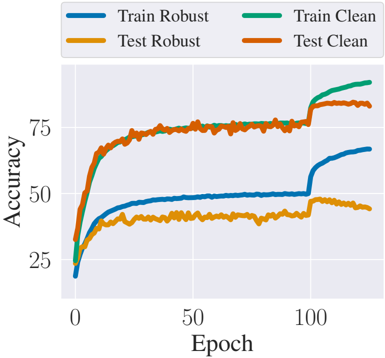
$$j^\star \in \operatorname*{arg\,max}_{j\in[K]-\{\mathbf{b}_i\}} h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}_{i,j^\star})_j - h_{\mathbf{x}}(\mathbf{a}_i + \boldsymbol{\delta}_{i,j^\star})_{\mathbf{b}_i}$$

Best paper award at ICML AdvML 2023

[1] https://infoscience.epfl.ch/record/302995 or https://tinyurl.com/33yup77v

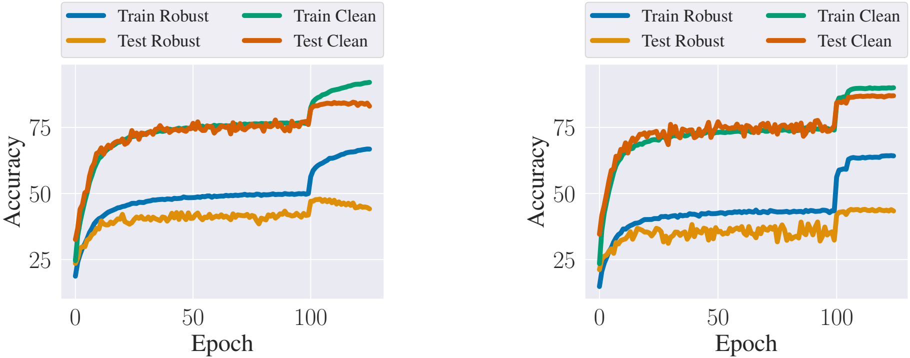
Practical Consequences of the Bilevel Formulation (I)

Figure: Learning curves of PGD^{10}-AT (Left) and BETA^{10}-AT



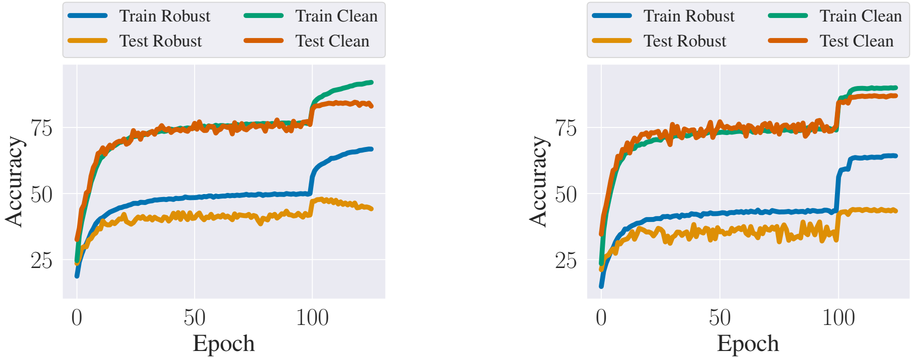
Practical Consequences of the Bilevel Formulation (I)

Figure: Learning curves of PGD^{10}-AT (Left) and BETA^{10}-AT (Right). Robust accuracy estimated with PGD^{20}



Practical Consequences of the Bilevel Formulation (I)

Figure: Learning curves of PGD^{10}-AT (Left) and BETA^{10}-AT (Right). Robust accuracy estimated with PGD^{20}



No Robust Overfitting occurs!

Practical Consequences of the Bilevel Formulation

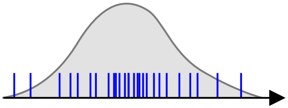
Table: Adversarial performance on CIFAR-10.

Training algorithm	Test accuracy											
	Clean		FGSM		PGD^{10}		PGD^{40}		$BETA^{10}$		APGD	
	Best	Last	Best	Last	Best	Last	Best	Last	Best	Last	Best	Last
FGSM	81.96	75.43	**94.26**	**94.22**	42.64	1.49	42.66	1.62	40.30	0.04	41.56	0.00
PGD^{10}	83.71	83.21	51.98	47.39	46.74	39.90	45.91	39.45	43.64	40.21	44.36	42.62
$TRADES^{10}$	81.64	81.42	52.40	51.31	47.85	42.31	47.76	42.92	44.31	40.97	43.34	41.33
$MART^{10}$	78.80	77.20	53.84	53.73	49.08	41.12	48.41	41.55	44.81	41.22	45.00	42.90
$BETA\text{-}AT^{5}$	**87.02**	**86.67**	51.22	51.10	44.02	43.22	43.94	42.56	42.62	42.61	41.44	41.02
$BETA\text{-}AT^{10}$	85.37	85.30	51.42	51.11	45.67	45.39	45.22	45.00	44.54	44.36	44.32	44.12
$BETA\text{-}AT^{20}$	82.11	81.72	54.01	53.99	**49.96**	**48.67**	**49.20**	**48.70**	**46.91**	**45.90**	**45.27**	**45.25**

Another minimax example: Generative adversarial networks (GANs)

∘ Ingredients:

- fixed *noise* distribution p_Ω (e.g., normal)
- target distribution $\hat{\mu}_n$ (natural images)
- $\mathcal{X}$ parameter class inducing a class of functions (generators)
- $\mathcal{Y}$ parameter class inducing a class of functions (dual variables)



Wasserstein GANs formulation [1]

Define a parameterized function $\mathrm{d}_{\mathbf{y}}(\mathbf{a})$, where $\mathbf{y} \in \mathcal{Y}$ such that $\mathrm{d}_{\mathbf{y}}(\mathbf{a})$ is 1-Lipschitz. In this case, the Wasserstein GAN training problem is given by

$$\min_{\mathbf{x}\in\mathcal{X}} \left(\max_{\mathbf{y}\in\mathcal{Y}} \boldsymbol{E}_{\mathbf{a}\sim\hat{\mu}_n} \left[\mathrm{d}_{\mathbf{y}}(\mathbf{a})\right] - \boldsymbol{E}_{\boldsymbol{\omega}\sim\mathrm{p}_\Omega} \left[\mathrm{d}_{\mathbf{y}}(h_{\mathbf{x}}(\boldsymbol{\omega}))\right] \right). \tag{1}$$

This problem is already captured by the template $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x}, \mathbf{y})$. Note that the original problem is a direct non-smooth minimization problem and the Rubinstein-Kantarovic duality results in the minimax template.

Remarks:
∘ Cannot solve in a manner similar to adversarial training a la Danskin. Need a direct approach.
∘ Scalability, mode collapse, catastrophic forgetting. Heuristics galore!
∘ Enforce Lipschitz constraint weight clipping, gradient penalty, spectral normalization [1, 9, 15].

Abstract minmax formulation

Minimax formulation

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{2}$$

where

- Φ is differentiable and nonconvex in $\mathbf{x}$ and nonconcave in $\mathbf{y}$,
- The domain is unconstrained, specifically $\mathcal{X} = \mathbb{R}^m$ and $\mathcal{Y} = \mathbb{R}^n$.

○ Key questions:

1. Where do the algorithms converge?
2. When do the algorithm converge?

Solving the minimax problem: Solution concepts

◦ Consider the unconstrained setting:

$$\boxed{\Phi^\star = \min_{\mathbf{x}} \max_{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})}$$

◦ Goal: Find an LNE point $(\mathbf{x}^\star, \mathbf{y}^\star)$.

Definition (Local Nash Equilibrium)

A pure strategy $(\mathbf{x}^\star, \mathbf{y}^\star)$ is called a local Nash equilibrium if

$$\Phi(\mathbf{x}^\star, \mathbf{y}) \leq \Phi(\mathbf{x}^\star, \mathbf{y}^\star) \leq \Phi(\mathbf{x}, \mathbf{y}^\star) \qquad \text{(LNE)}$$

for all $\mathbf{x}$ and $\mathbf{y}$ within some neighborhood of $\mathbf{x}^\star$ and $\mathbf{y}^\star$, i.e., $\|\mathbf{x} - \mathbf{x}^\star\| \leq \varepsilon$ and $\|\mathbf{y} - \mathbf{y}^\star\| \leq \varepsilon$ for some $\varepsilon > 0$.

Abstract minmax formulation

Minimax formulation

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y}), \tag{3}$$

where

- Φ is differentiable and nonconvex in $\mathbf{x}$ and nonconcave in $\mathbf{y}$,
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A buffet of negative results [6]

"Even when the objective is a Lipschitz and smooth differentiable function, deciding whether a min-max point exists, in fact even deciding whether an approximate min-max point exists, is NP-hard. More importantly, an approximate local min-max point of large enough approximation is guaranteed to exist, but finding one such point is PPAD-complete. The same is true of computing an approximate fixed point of the (Projected) Gradient Descent/Ascent update dynamics."

Basic algorithms for minimax

- Given $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$, define $V(\mathbf{z}) = [\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), -\nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]$ with $\mathbf{z} = [\mathbf{x},\mathbf{y}]$.

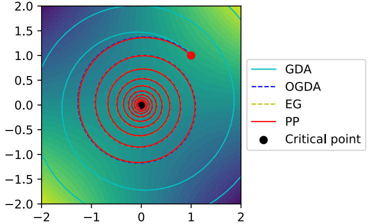

Figure: Trajectory of different algorithms for a simple bilinear game $\min_x \max_y xy$.

- (In)Famous algorithms
  - Gradient Descent Ascent (GDA)
  - Proximal point method (PPM) [18, 8]
  - Extra-gradient (EG) [12]
  - Optimistic GDA (OGDA) [19, 14]
  - Reflected-Forward-Backward-Splitting (RFBS) [4]

- EG and OGDA are approximations of the PPM
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha V(\mathbf{z}^k)$.
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  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha V(\mathbf{z}^k - \alpha V(\mathbf{z}^k))$.
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha[2V(\mathbf{z}^k) - V(\mathbf{z}^{k-1})]$.
  - $\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha V(2\mathbf{z}^k - \mathbf{z}^{k-1})$.

Where do the algorithms converge?

- Recall: Given $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{x},\mathbf{y})$, define $V(\mathbf{z}) = [\nabla_{\mathbf{x}}\Phi(\mathbf{x},\mathbf{y}), -\nabla_{\mathbf{y}}\Phi(\mathbf{x},\mathbf{y})]$ with $\mathbf{z} = [\mathbf{x},\mathbf{y}]$.
- Given $V(\mathbf{z})$, define stochastic estimates of $V(\mathbf{z},\zeta) = V(\mathbf{z}) + U(\mathbf{z},\zeta)$, where
  - $U(\mathbf{z},\zeta)$ is a bias term,
  - We often have unbiasedness: $\boldsymbol{E}U(\mathbf{z},\zeta) = 0$,
  - The bias term can have bounded moments,
  - We often have bounded variance: $P(\|U(\mathbf{z},\zeta)\| \geq t) \leq 2\exp -\frac{t^2}{2\sigma^2}$ for $\sigma > 0$.
- An abstract template for generalized Robbins-Monro schemes, dubbed as $\mathcal{A}$:

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \alpha_k V(\mathbf{z}^k, \zeta^k).$$

The dessert section in the buffet of negative results: [10]

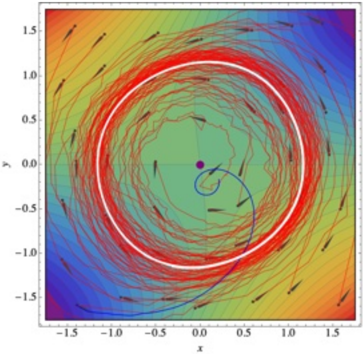
1. Bounded trajectories of $\mathcal{A}$ always converge to an internally chain-transitive (ICT) set.
2. Trajectories of $\mathcal{A}$ may converge with arbitrarily high probability to spurious attractors that contain no critical point of Φ.

Minimax is more difficult than just optimization [10]

- Internally chain-transitive (ICT) sets characterize the convergence of dynamical systems [2].
  - For optimization, {attracting ICT} $\equiv$ {solutions}
  - For minimax, {attracting ICT} $\equiv$ {solutions} $\cup$ {spurious sets}

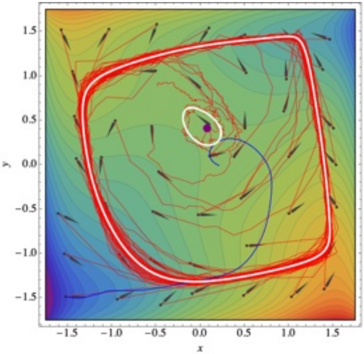
- "Almost" bilinear $\neq$ bilinear:

$$\Phi(x, y) = xy + \epsilon\phi(x), \phi(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$$



- The "forsaken" solutions:

$$\Phi(y, x) = y(x-0.5) + \phi(y) - \phi(x), \phi(u) = \frac{1}{4}u^2 - \frac{1}{2}u^4 + \frac{1}{6}u^6$$



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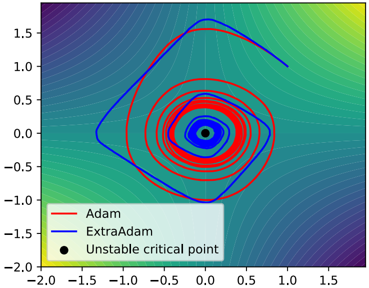

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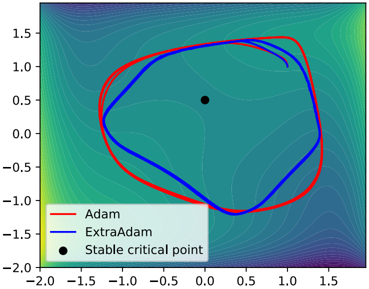


When do the algorithms converge?

Assumption (weak Minty variational inequality)

For some $\rho \in \mathbb{R}$, weak MVI implies

$$\langle V(\mathbf{z}), \mathbf{z} - \mathbf{z}^\star \rangle \geqslant \rho \|V(\mathbf{z})\|^2, \quad \text{for all } \mathbf{z} \in \mathbb{R}^n. \tag{4}$$

- A variant EG+ converges when $\rho > -\frac{1}{8L}$
  - Diakonikolas, Daskalakis, Jordan, AISTATS 2021.
- It still cannot handle the examples of [10].

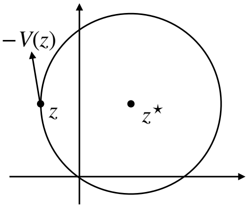
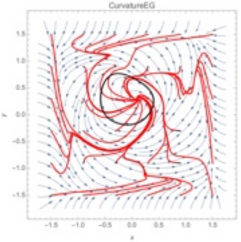


Figure: The operator $V(z)$ is allowed to point away from the solution by some amount when ρ is negative.

- Complete picture under weak MVI (ICLR'22 and '23)
  - Pethick, Lalafat, Patrinos, Fercoq, and Cevher.
  - constrained and regularized settings with $\rho > -\frac{1}{2L}$
  - matching lower bounds
  - stochastic variants handling the examples of [10]
  - adaptive variants handling the examples of [10]



GANs with SEG+ [17]

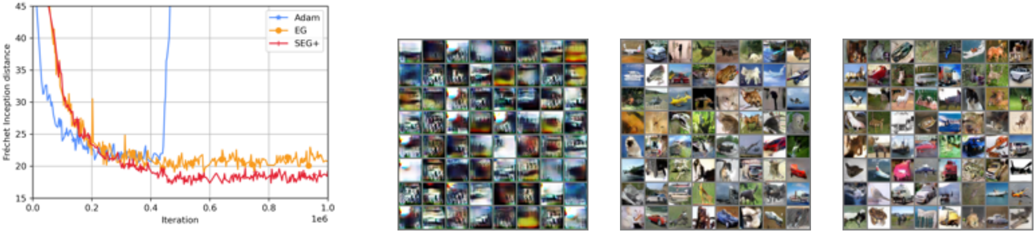


Figure: A performance comparison of GAN training by Adam, EG with stochastic gradients, and SEG+.

Robustness of the worst-performing class [16]

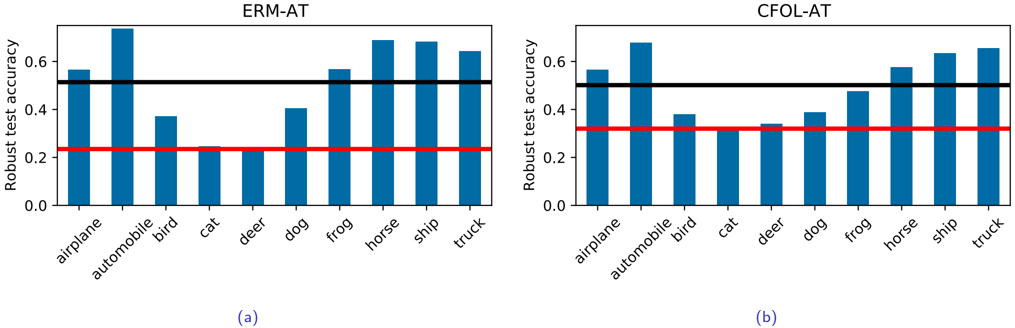
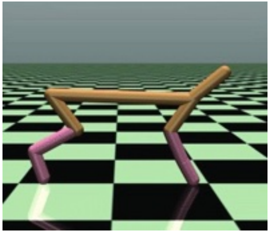


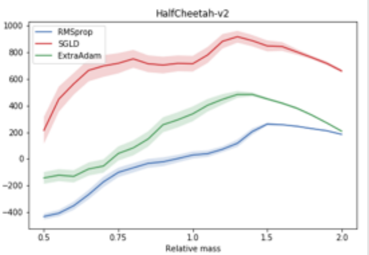
Figure: Robust test accuracy of (a) Empirical Risk Minimization and (b) the class focused online learning.

Code: `https://github.com/LIONS-EPFL/class-focused-online-learning-code`

Take home messages

- Even the simplified view of robust & adversarial ML is challenging
- $\min$-$\max$-type has spurious attractors with no equivalent concept in $\min$-type
- Not all step-size schedules are considered in our work: Possible to "converge" under some settings
- Other successful attempts[1] consider "mixed Nash" concepts[2]





- Existing theory and methods for adversarial training is wrong!

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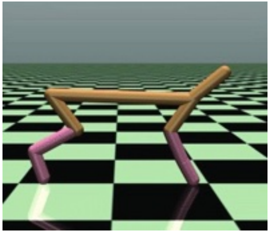

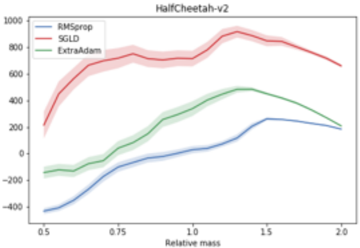

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